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WAVE REGIMES ON A NONISOTHERMAL FILM OF A VISCOUS LIQUID FLOWING DOWN A VERTICAL PLANE

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A thin film flow of a viscous liquid flowing down a vertical wall in the field of the gravity force is studied. The values of temperatures on the solid wall and on the free surface are constant. The viscosity and thermal diffusivity are functions of temperature. An equation that describes the evolution of surface disturbances is derived for small flow rates in the long-wave approximation.

Key words: *nonisothermal film flow, evolution equation, soliton solutions.*

Thin liquid films flowing down in the field of the gravity force have been studied for more than 50 years, because this phenomenon is widely used in various technological processes.

Considering the behavior of isothermal liquid films depending on parameters of the initial undisturbed flow provided a rather large number of models that have to be studied. Nonisothermal films have aroused significant recent interest. Allowance for the effect of temperature is a complicating factor. As the temperature affects the physical parameters of transport, local heating and cooling support thermocapillary effects, and liquid condensation or evaporation affect the flow geometry and conditions on the free surface, many of the models are multiparametric, and it is difficult to analyze them in detail (see, e.g., [1, 2]).

In the present work, we consider a viscous liquid film flow down a vertical wall in the field of the gravity force. The density and specific heat are assumed to be independent of temperature. The values of temperature on the solid wall and on the free surface are maintained constant. The viscosity and thermal diffusivity are certain functions of temperature. Such an approach allows adequate modeling of film flows with evaporation or condensation, where the heat fluxes are not too intense and there are minor changes in film thickness.

The main objective of this work was to derive a model equation that could be used to study wave regimes of a nonisothermal liquid film flow.

1. Formulation of the Problem. We consider a thin film flow of a viscous liquid down a vertical plane in the field of the gravity force. The flow structure and the coordinate system used are shown in Fig. 1.

The dependence of the density ρ and specific heat c of the liquid on temperature is neglected. The values of temperature on the solid wall T_w and on the free surface T_s are maintained constant. The viscosity μ and thermal diffusivity a are certain functions of temperature:

$$\mu = \mu_0 \varphi(\theta), \qquad a = a_0 f(\theta).$$

Here μ_0 and a_0 are the values of viscosity and thermal diffusivity on the free boundary; $\theta = (T - T_w)/(T_s - T_w)$.

The isothermal viscous liquid film flow of constant thickness is known to be unstable to infinitesimal perturbations even in the case of extremely small flow rates; further evolution of perturbations leads to formation of waves. To describe similar regimes in a nonisothermal viscous film, we write the equations of motion in dimensionless form. Let h_0 be the film thickness without perturbations, U_0 be the velocity on the free surface, and L be the characteristic longitudinal scale of the perturbation. Then, using the quantities L/U_0 and U_0 as scales of time and velocity, $\mu_0 U_0/h_0$ as a scale of stress tensor components, $\rho g h_0$ as a scale of pressure, and L and h_0 as scales in x, z, and y directions, respectively, we obtain the problem in dimensionless variables

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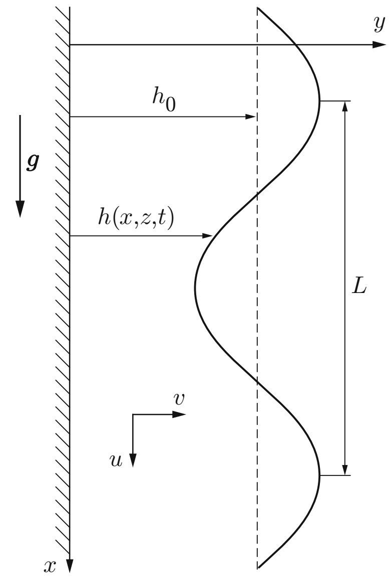

Fig. 1. Flow structure.

$$\varepsilon\,\frac{\partial u}{\partial t}+\varepsilon u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial y}+\varepsilon w\,\frac{\partial u}{\partial z}=\frac{1}{\mathrm{Fr}}\Big(1-\varepsilon\frac{\partial p}{\partial x}\Big)+\frac{1}{\mathrm{Re}}\Big(\varepsilon\,\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\varepsilon\,\frac{\partial\tau_{xz}}{\partial z}\Big),$$

$$\varepsilon\,\frac{\partial v}{\partial t}+\varepsilon u\,\frac{\partial v}{\partial x}+v\,\frac{\partial v}{\partial y}+\varepsilon w\,\frac{\partial v}{\partial z}=-\frac{1}{\mathrm{Fr}}\,\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\Big(\varepsilon\,\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\varepsilon\,\frac{\partial\tau_{yz}}{\partial z}\Big),$$

$$\varepsilon\,\frac{\partial w}{\partial t}+\varepsilon u\,\frac{\partial w}{\partial x}+v\,\frac{\partial w}{\partial y}+\varepsilon w\,\frac{\partial w}{\partial z}=-\frac{\varepsilon}{\mathrm{Fr}}\,\frac{\partial p}{\partial z}+\frac{1}{\mathrm{Re}}\Big(\varepsilon\,\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\varepsilon\,\frac{\partial\tau_{zz}}{\partial z}\Big),\qquad(1.1)$$

$$\varepsilon\,\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\varepsilon\,\frac{\partial w}{\partial z}=0,$$

$$\varepsilon\,\frac{\partial\theta}{\partial t}+\varepsilon u\,\frac{\partial\theta}{\partial x}+v\,\frac{\partial\theta}{\partial y}+\varepsilon w\,\frac{\partial\theta}{\partial z}=\frac{1}{\mathrm{Pe}}\Big(\varepsilon^2\,\frac{\partial}{\partial x}\Big(f\,\frac{\partial\theta}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(f\,\frac{\partial\theta}{\partial y}\Big)+\varepsilon^2\,\frac{\partial}{\partial z}\Big(f\,\frac{\partial\theta}{\partial z}\Big)\Big)$$

with the conditions on the solid wall ($y=0$) and on the free surface [$y=h(x,z,t)$] in the form

$$u=v=w=\theta=0,\qquad y=0,$$

$$(p-p_0-\mathrm{We}\,(K_1+K_2))n_i-(\mathrm{Fr}\,/\,\mathrm{Re})\tau_{ik}n_k=0,\qquad y=h(x,z,t),\qquad(1.2)$$

$$\theta=1.$$

Here u, v, and w are the x, y, and z components of velocity, p is the pressure in the liquid, p_0 is the ambient pressure (without loss of generality, we can assume that $p_0=0$), n_i are the components of the normal vector

$$\boldsymbol{n}=\frac{(-\varepsilon h_x,1,-\varepsilon h_z)}{\sqrt{1+\varepsilon^2h_x^2+\varepsilon^2h_z^2}},$$

K_i are the dimensionless principal curvatures of the free surface

$$K_1+K_2=-\frac{(1+\varepsilon^2h_x^2)\varepsilon h_{zz}-2\varepsilon^3h_xh_zh_{xz}+(1+\varepsilon^2h_z^2)\varepsilon h_{xx}}{(1+\varepsilon^2h_x^2+\varepsilon^2h_z^2)^{3/2}},$$

(the subscripts at the quantity h mean differentiation with respect to the indicated variable), and τ_{ik} are the dimensionless components of the stress tensor:

$$\tau_{xx}=2\varphi(\theta)\varepsilon\,\frac{\partial u}{\partial x},\qquad\tau_{yy}=2\varphi(\theta)\,\frac{\partial v}{\partial y},\qquad\tau_{zz}=2\varphi(\theta)\varepsilon\,\frac{\partial w}{\partial z},$$

$$\tau_{xy} = \tau_{yx} = \varphi(\theta)\Big(\frac{\partial u}{\partial y} + \varepsilon\,\frac{\partial v}{\partial x}\Big), \qquad \tau_{xz} = \tau_{zx} = \varphi(\theta)\varepsilon\Big(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\Big), \tag{1.3}$$

$$\tau_{yz} = \tau_{zy} = \varphi(\theta)\Big(\varepsilon\,\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\Big).$$

Equations (1.1)–(1.3) contain the following parameters: $\varepsilon = h_0/L$, Reynolds number $\mathrm{Re} = \rho h_0 U_0/\mu_0$, Froude number $\mathrm{Fr} = U_0^2/(gh_0)$, Weber number $\mathrm{We} = \sigma/(\rho g h_0^2)$, and Peclet number $\mathrm{Pe} = h_0 U_0/a_0$.

The following kinematic condition is also satisfied on the free boundary:

$$\varepsilon\Big(\frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x} + w\,\frac{\partial h}{\partial z}\Big) = v, \qquad y = h(x, z, t). \tag{1.4}$$

If there are no perturbations, problem (1.1)–(1.3) reduces to the following system of equations:

$$\frac{1}{\mathrm{Fr}} + \frac{1}{\mathrm{Re}}\,\frac{d}{dy}\Big(\varphi(\theta_0)\,\frac{dU}{dy}\Big) = 0, \qquad \frac{d}{dy}\Big(f(\theta_0)\,\frac{d\theta_0}{dy}\Big) = 0; \tag{1.5}$$

$$U = \theta_0 = 0, \qquad y = 0,$$

$$U = \theta_0 = 1, \qquad \tau_{xy} = \varphi(\theta_0)\,\frac{dU}{dy} = 0, \qquad y = 1. \tag{1.6}$$

To solve this problem, we have to know particular forms of the functions $\varphi(\theta_0)$ and $f(\theta_0)$. The solution of the system is written as

$$U(y) = \varphi_1(y)/\varphi_1(1).$$

The temperature profile θ_0 is determined as an implicit function

$$y = f_1(\theta_0)/f_1(1).$$

Here

$$\varphi_1(y) = \int\limits_0^y \frac{(1 - y)\,dy}{\varphi(\theta_0)}, \qquad f_1(\theta) = \int\limits_0^\theta f(\theta)\,d\theta.$$

By virtue of normalization $U(1) = 1$, we have $\mathrm{Fr}/\mathrm{Re} = \varphi_1(1)$.

In what follows, we confine ourselves to considering long-wave disturbances (i.e., $\varepsilon \ll 1$) and assume that the Reynolds numbers are sufficiently low: $\mathrm{Re} \simeq 1$.

To use the method of multiple scales (see, e.g., [3]), we introduce a set of fast and slow times and new functions:

$$\tau_m = \varepsilon^m t, \qquad m = 0, 1, 2, \ldots,$$

$$u = U + \varepsilon u', \quad v = \varepsilon^2 v', \quad w = \varepsilon w', \quad p = p_0 + \varepsilon p', \quad \theta = \theta_0 + \varepsilon\theta', \quad h = 1 + \varepsilon h'.$$

In further consideration of the problem, we neglect terms of high orders with respect to ε. Thus, the functions $\varphi(\theta)$ and $f(\theta)$ can be presented as

$$\varphi = \varphi_0 + \varepsilon\varphi_0'\theta' + \varepsilon^2\varphi_0''\theta'^2/2, \qquad f = f_0 + \varepsilon f_0'\theta' + \varepsilon^2 f_0''\theta'^2/2.$$

Here $\varphi_0 = \varphi(\theta_0)$, $\varphi_0' = d\varphi(\theta)/d\theta$ for $\theta = \theta_0$, $\varphi_0'' = d\varphi_0'/d\theta$ for $\theta = \theta_0$, $f_0 = f(\theta_0)$, $f_0' = df(\theta)/d\theta$ for $\theta = \theta_0$, and $f_0'' = df_0'/d\theta$ for $\theta = \theta_0$. Thus, the values of the functions $\varphi(\theta)$ and $f(\theta)$ and their derivatives are taken for values of the functional argument corresponding to a plane film flow described by problem (1.5), (1.6).

Neglecting terms of the order of ε^2 and higher and shifting the boundary conditions from the free surface to its undisturbed level (i.e., expanding all functions with respect to the powers of $\varepsilon h'$), we obtain the system of equations (with omitted primes at the disturbed quantities)

$$\varepsilon\Big(\frac{\partial u}{\partial\tau_0} + U\,\frac{\partial u}{\partial x} + v\,\frac{dU}{dy} + \frac{1}{\mathrm{Fr}}\,\frac{\partial p}{\partial x}\Big) = \frac{1}{\mathrm{Re}}\,\frac{\partial}{\partial y}\Big[\varphi_0\frac{\partial u}{\partial y} + \varphi_0'\,\frac{dU}{dy}\,\theta + \varepsilon\Big(\varphi_0'\theta\,\frac{\partial u}{\partial y} + \frac{1}{2}\,\varphi_0''\theta^2\,\frac{dU}{dy}\Big)\Big],$$

$$\frac{1}{\mathrm{Fr}}\,\frac{\partial p}{\partial y} = \frac{\varepsilon}{\mathrm{Re}}\Big[\varphi_0\,\frac{\partial^2 u}{\partial x\,\partial y} + \varphi_0'\,\frac{dU}{dy}\,\frac{\partial\theta}{\partial x} + \varphi_0\,\frac{\partial^2 w}{\partial y\,\partial z} + 2\,\frac{\partial}{\partial y}\Big(\varphi_0\,\frac{\partial v}{\partial y}\Big)\Big],$$

$$\varepsilon\Big(\frac{\partial w}{\partial \tau_0} + U\,\frac{\partial w}{\partial x} + \frac{1}{\mathrm{Fr}}\,\frac{\partial p}{\partial z}\Big) = \frac{1}{\mathrm{Re}}\Big[\frac{\partial}{\partial y}\Big(\varphi_0\,\frac{\partial w}{\partial y}\Big) + \varepsilon\,\frac{\partial}{\partial y}\Big(\varphi_0'\theta\,\frac{\partial w}{\partial y}\Big)\Big], \tag{1.7}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

$$\varepsilon\Big(\frac{\partial \theta}{\partial \tau_0} + U\,\frac{\partial \theta}{\partial x} + v\,\frac{d\theta_0}{dy}\Big) = \frac{1}{\mathrm{Pe}}\Big[\frac{\partial}{\partial y}\Big(f_0\,\frac{\partial \theta}{\partial y} + f_0'\theta\,\frac{d\theta_0}{dy}\Big) + \varepsilon\frac{\partial}{\partial y}\Big(f_0'\theta\,\frac{\partial \theta}{\partial y} + \frac{1}{2}\,f_0''\,\theta^2\,\frac{d\theta_0}{dy}\Big)\Big]$$

with the boundary conditions

$$u = v = w = \theta = 0, \qquad y = 0,$$

$$\frac{\partial u}{\partial y} - \frac{h}{\varphi_1} + \varepsilon\Big[\varphi_0'\theta\,\frac{\partial u}{\partial y} + h\,\frac{\partial}{\partial y}\Big(\varphi_0'\theta\,\frac{dU}{dy} + \varphi_0\,\frac{\partial u}{\partial y}\Big)\Big] = 0, \qquad y = 1,$$

$$p + \varepsilon\,\frac{\partial p}{\partial y}\,h + \mathrm{We}\,\varepsilon^2\Delta h - 2\varepsilon\,\frac{\mathrm{Fr}}{\mathrm{Re}}\,\frac{\partial v}{\partial y} = 0, \qquad y = 1, \tag{1.8}$$

$$\frac{\partial w}{\partial y} + \varepsilon\Big[h\,\frac{\partial}{\partial y}\Big(\varphi_0\,\frac{\partial w}{dy}\Big) + \varphi_0'\theta\,\frac{\partial w}{\partial y}\Big] = 0, \qquad \theta + h\Big(\frac{d\theta_0}{dy} + \varepsilon\,\frac{\partial \theta}{dy}\Big) = 0, \qquad y = 1.$$

In writing condition (1.8), we take into account that $\varphi_0(1) = f_0(1) = 1$ and the Laplace operator is $\Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2$. Terms of higher orders with respect to ε are also left in Eq. (1.8), because the values of We for thin liquid films are normally rather high. Therefore, we assume that $\mathrm{We} \gg 1$ and $\mathrm{We}\,\varepsilon^2 \simeq 1$.

The kinematic condition (1.4) acquires the form

$$h_{\tau_0} + \varepsilon h_{\tau_1} + h_x + \varepsilon u h_x + \varepsilon w h_z = v + \varepsilon\,\frac{\partial v}{\partial y}\,h, \qquad y = 1. \tag{1.9}$$

The solution of problem (1.7), (1.8) is sought in the form of series with respect to ε:

$$(u, v, w, p, \theta, h) = \sum_{m=0}^{\infty} \varepsilon^m (u^m, v^m, w^m, p^m, \theta^m, h^m). \tag{1.10}$$

Equating the coefficients at identical powers of ε in the original system of equations to zero, we obtain simpler [than Eqs. (1.7) and (1.8)] systems corresponding to different orders of ε. The quantities u^m, v^m, w^m, and θ^m as functions of h^m can be easily obtained from these systems. Substituting these data into the kinematic condition (1.9), we obtain the equation for film thickness perturbations. Thus, for a zero order, Eqs. (1.10), (1.7), and (1.8) yield the system

$$\frac{\partial}{\partial y}\Big(\varphi_0\,\frac{\partial u^0}{\partial y} + \varphi_0'\,\frac{dU}{dy}\,\theta^0\Big) = 0, \qquad \frac{\partial p^0}{\partial y} = 0, \qquad \frac{\partial}{\partial y}\Big(\varphi_0\,\frac{\partial w^0}{\partial y}\Big) = 0,$$

$$\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} + \frac{\partial w^0}{\partial z} = 0, \qquad \frac{\partial}{\partial y}\Big(f_0\,\frac{\partial \theta^0}{\partial y} + f_0'\,\frac{d\theta_0}{dy}\,\theta^0\Big) = 0 \tag{1.11}$$

with the boundary conditions

$$u^0 = v^0 = w^0 = \theta^0 = 0, \qquad y = 0,$$

$$\frac{\partial u^0}{\partial y} - \frac{h^0}{\varphi_1} = 0, \qquad p^0 + \mathrm{We}\,\varepsilon^2\Delta h^0 = 0, \qquad \frac{\partial w^0}{dy} = 0, \qquad \theta^0 + f_1 h^0 = 0, \qquad y = 1. \tag{1.12}$$

Solving problem (1.11), (1.12), we obtain the following expressions for the disturbed quantities of the zero order:

$$u^0(x, y, z, t) = (\mathrm{Re}\,/\,\mathrm{Fr})\varphi_2 h^0, \qquad v^0(x, y, z, t) = -(\mathrm{Re}\,/\,\mathrm{Fr})\varphi_3 h^0_x,$$

$$w^0(x, y, z, t) = 0, \qquad p^0 = -\,\mathrm{We}\,\varepsilon^2\Delta h^0, \qquad \theta^0 = F_1 h^0. \tag{1.13}$$

Here

$$F_1 = -\frac{f_1(1)y}{f_0(\theta_0(y))}, \quad \varphi_2 = \int\limits_0^y \Big(1 - \frac{\varphi_0'}{\varphi_0} F_1(y)(1-y)\Big) \frac{dy}{\varphi_0(\theta^0(y))}, \quad \varphi_3 = \int\limits_0^y \varphi_2(y)\, dy.$$

Substituting Eq. (1.13) into Eq. (1.9), we obtain an equation that describes the behavior of disturbances in the first approximation:

$$h^0_{\tau_0} + c_0 h^0_x = 0, \qquad c_0 = 1 + \varphi_3(1)/\varphi_1(1). \tag{1.14}$$

It follows from here that, in the first approximation, all disturbances propagate with a constant velocity, which is greater than the flow velocity on the flat free boundary by a factor of c_0:

$$h^0 = h^0(\xi), \qquad \xi = x - c_0\tau_0.$$

For the next order with respect to ε, Eqs. (1.10), (1.7), and (1.8) yield the system

$$\frac{\partial u^0}{\partial \tau_0} + U\,\frac{\partial u^0}{\partial x} + v^0\,\frac{dU}{dy} + \frac{1}{\mathrm{Fr}}\,\frac{\partial p^0}{\partial x}$$

$$= \frac{1}{\mathrm{Re}}\,\frac{\partial}{\partial y}\Big(\varphi_0\,\frac{\partial u^1}{\partial y} + \varphi_0'\,\frac{dU}{dy}\,\theta^1 + \varphi_0'\theta^0\,\frac{\partial u^0}{\partial y} + \frac{1}{2}\,\varphi_0''\,\frac{dU}{dy}\,(\theta^0)^2\Big),$$

$$\frac{1}{\mathrm{Fr}}\,\frac{\partial p^1}{\partial y} = \frac{1}{\mathrm{Re}}\Big[\varphi_0\,\frac{\partial^2 u^0}{\partial x\,\partial y} + \varphi_0'\,\frac{dU}{dy}\,\frac{\partial \theta^0}{\partial x} + 2\,\frac{\partial}{\partial y}\Big(\varphi_0\,\frac{\partial v^0}{\partial y}\Big)\Big],$$

$$\frac{1}{\mathrm{Fr}}\,\frac{\partial p^0}{\partial z} = \frac{1}{\mathrm{Re}}\,\frac{\partial}{\partial y}\Big(\varphi_0\,\frac{\partial w^1}{\partial y}\Big), \tag{1.15}$$

$$\frac{\partial \theta^0}{\partial \tau_0} + U\,\frac{\partial \theta^0}{\partial x} + v^0\,\frac{d\theta_0}{dy} = \frac{1}{\mathrm{Pe}}\,\frac{\partial}{\partial y}\Big(f_0\,\frac{\partial \theta^1}{\partial y} + f_0'\theta^1\,\frac{d\theta_0}{dy} + f_0'\theta^0\,\frac{\partial \theta^0}{\partial y} + \frac{1}{2}\,f_0''\,\frac{d\theta_0}{dy}\,(\theta^0)^2\Big),$$

$$u^1 = v^1 = w^1 = \theta^1 = 0, \qquad y = 0,$$

$$\frac{\partial u^1}{\partial y} - \frac{h^1}{\varphi_1} + \varphi_0'\theta^0\,\frac{\partial u^0}{\partial y} = 0, \qquad p^1 + \mathrm{We}\,\varepsilon^2\Delta h^1 = 2\,\frac{\mathrm{Fr}}{\mathrm{Re}}\,\frac{\partial v^0}{\partial y}, \qquad y = 1,$$

$$\frac{\partial w^1}{dy} = 0, \qquad \theta^1 + f_1 h^1 + \frac{\partial \theta^0}{\partial y}\,h^0 + \frac{1}{2}\,\frac{d^2\theta_0}{dy^2}\,(h^0)^2 = 0, \qquad y = 1.$$

Solving problem (1.15) and taking into account Eqs. (1.13) and (1.14), we obtain the following formulas for disturbances of this order:

$$w^1(x,y,z,t) = \mathrm{We}\,\varepsilon^2 U(y)\,\Delta h^0_z,$$

$$u^1(x,y,z,t) = \mathrm{Re}\,\varphi_7 h^0_x + \mathrm{We}\,\varepsilon^2 U(y)\,\Delta h^0_x + \varphi_8 (h^0)^2 + (\varphi_2/\varphi_1(1))h^1, \tag{1.16}$$

$$v^1(x,y,z,t) = -(\mathrm{Re}\,\varphi_9 h^0_{xx} + \mathrm{We}\,\varepsilon^2\varphi_{10}\,\Delta^2 h^0 + 2\varphi_{11} h^0 h^0_x + (\varphi_3/\varphi_1(1))h^1_x),$$

$$\theta^1 = \mathrm{Pe}\,F_2 h^0_x + F_3 (h^0)^2 + F_1 h^1.$$

In Eqs. (1.16), we do not give terms for the disturbances h^1, $v^1(x,y,z,t)$, and $u^1(x,y,z,t)$, with respect to which the kinematic condition is invariant. The functions F_i and φ_i in these expressions are determined as follows:

$$F_2 = \frac{1}{f_0(\theta^0(y))}\Big(\int\limits_0^y \varphi_4\, dy - y\int\limits_0^1 \varphi_4\, dy\Big), \qquad \varphi_4 = \int\limits_0^y \Big((U - c_0)F_1 - \frac{\varphi_3 f_1(1)}{\varphi_1(1) f_0}\Big)\, dy,$$

$$F_3 = \frac{1}{f_0(\theta^0(y))}\Big(\int\limits_0^y \varphi_5\, dy - y\int\limits_0^1 \varphi_5\, dy - f_2(1)y\Big),$$

$$f_2 = f_1(1)\Big[1 + \frac{d}{dy}\Big(\frac{1}{2f_0}\Big)\Big], \qquad \varphi_5 = -\frac{1}{2}\Big(f_0'\,\frac{d(F_1^2)}{dy} + f_1(1)F_1^2\,\frac{f_0''}{f_0}\Big),$$

$$\varphi_6(y) = \frac{1}{\varphi_1(1)}\Big[\frac{1}{\varphi_0(\theta^0(y))}\int\limits_0^y \Big((U - c_0)\varphi_2 - \varphi_3\,\frac{dU}{dy}\Big)\,dy\Big] - \frac{\mathrm{Pe}}{\mathrm{Re}}\,F_2\,\frac{\varphi_0'}{\varphi_0}\,\frac{dU}{dy},$$

$$\varphi_7 = \int\limits_0^y \varphi_6(y)\,dy - \varphi_6(1)\int\limits_0^y \frac{dy}{\varphi_0(y)},$$

$$\varphi_8 = -\int\limits_0^y \frac{dy}{\varphi_0(y)}\Big[\Big(\varphi_0' F_3 + \frac{1}{2}\,\varphi_0'' F_1^2\Big)\,\frac{dU}{dy} + F_1\,\frac{\varphi_0'}{\varphi_1(1)}\,\frac{dF_2}{dy}\Big],$$

$$\varphi_9 = \int\limits_0^y \varphi_7(y)\,dy, \qquad \varphi_{10} = \int\limits_0^y U(y)\,dy, \qquad \varphi_{11} = \int\limits_0^y \varphi_8(y)\,dy.$$

2. Model Equation. Substituting Eqs. (1.16) into the kinematic condition (1.9) and requiring the absence of secular terms in the expression for h^1, we obtain a nonlinear equation for determining h^0:

$$h^0_{\tau_1} + Ah^0h^0_x + \mathrm{Re}\,\varphi_9(1)h^0_{xx} + \mathrm{We}\,\varepsilon^2\varphi_{10}(1)\,\Delta^2 h^0 = 0. \qquad (2.1)$$

Here $A = 2\varphi_{11}(1) + \varphi_2(1)/\varphi_1(1)$.

Knowing h^0, we can determine all the remaining disturbed quantities in Eq. (1.10) up to the first order inclusive from Eqs. (1.13) and (1.16).

Thus, Eq. (2.1) describes the evolution of spatial disturbances on a nonisothermal liquid film flowing down a vertical plane. It follows from Eq. (1.14) that Eq. (2.1) is written in a frame of reference moving with a velocity c_0 with respect to the wall.

Let us specify the choice of the characteristic longitudinal scale L. We require the absolute values of the coefficients at the third and fourth terms in Eq. (2.1) to be identical. Then, for the small parameter ε used in the expansion, we obtain

$$\varepsilon = \sqrt{\frac{\mathrm{Re}\,|\varphi_9(1)|}{\mathrm{We}\,\varphi_{10}(1)}}\,;$$

correspondingly, the characteristic longitudinal size of disturbances is determined by the equality

$$L = \sqrt{\frac{\mathrm{We}\,\varphi_{10}(1)}{\mathrm{Re}\,|\varphi_9(1)|}}\,h_0.$$

It follows from these relations that the assumption of the long-wave character of disturbances considered is valid for high Weber numbers, like in the case of isothermal liquid films. In addition, the ratio $\varphi_{10}(1)/|\varphi_9(1)|$ should be of the order of unity. Here we take into account that the dimensionless flow rate in the film is $\varphi_{10}(1) > 0$. Finally, using the substitution

$$h = aH, \quad \tau = d\tau_1, \quad a = \mathrm{Re}\,\varphi_9(1)/(4A), \quad d = \mathrm{Re}\,\varphi_9(1), \qquad (2.2)$$

we transform Eq. (2.1) to the equation

$$\frac{\partial H}{\partial \tau} + 4H\,\frac{\partial H}{\partial x} + \mathrm{sgn}\,\varphi_9(1)\,\frac{\partial^2 H}{\partial x^2} + \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\Big)^2 H = 0. \qquad (2.3)$$

Equation (2.3) is a typical example of model equations arising in studying the evolution of disturbances in active dissipative media, where infinitesimal periodic pertubations exponentially grow or decay, depending on the value of the wavenumber. Depending on the sign of $\varphi_9(1)$, Eq. (2.3) has two forms:

$$\frac{\partial H}{\partial \tau} + 4H\,\frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2} + \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\Big)^2 H = 0; \qquad (2.4)$$

$$\frac{\partial H}{\partial \tau} + 4H\,\frac{\partial H}{\partial x} - \frac{\partial^2 H}{\partial x^2} + \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\Big)^2 H = 0. \tag{2.5}$$

The signs in Eq. (2.3) can be determined only if the form of the temperature dependences of the viscosity μ and thermal diffusivity a are known: $\mu = \mu_0 \varphi(\theta)$ and $a = a_0 f(\theta)$.

Note that Eqs. (2.4) and (2.5) for plane waves $H = H(x, \tau)$ coincide with the equations obtained in [4] in studying plane disturbances on a two-layer isothermal film. It was noted [4] that the one-dimensional analog of Eq. (2.4) is of most interest from the viewpoint of realization of rather complex wave solutions. In this case, Eq. (2.4) has the form

$$\frac{\partial H}{\partial \tau} + 4H\,\frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2} + \frac{\partial^4 H}{\partial x^4} = 0. \tag{2.6}$$

This equation is currently known as the Kuramoto–Sivashinsky equation. As applied to film flows, this equation was first derived in [5]. Equation (2.6) was studied in much detail, and many of its solutions were found (see, e.g., [6, 7]).

In the case of Eq. (2.5), the structure of plane solutions is simple: they all decay with time. We can easily demonstrate that the spatial solutions of Eq. (2.5) behave in the same manner. Of greatest interest is the case where the model yields Eq. (2.4). It is this equation that is obtained for isothermal films [8]. Apparently, an identical equation will be obtained for typical dependences of μ and a on θ.

Thus, within the framework of approximations used for different dependences of μ and a on θ, the description of the behavior of spatial disturbances reduces to studying the solutions of one equation (2.3). The wave profiles are similar and are determined by the transformation parameters (2.2), i.e., their characteristic velocities c_0 and the coefficients in Eq. (2.1) depend on the form of μ and a, but the waves themselves are "topologically" similar: particular forms of the wave profiles for different models can be obtained one from another by simple recalculations. Thus, the currently available information on the wave solutions of Eq. (2.4) in studying the wave regimes of the isothermal film flow offers an idea of the wave pattern of nonisothermal flows. For instance, a systematic study of the steadily traveling solutions

$$H = H(\xi, z), \qquad \xi = x - c\tau$$

showed that Eq. (2.4) has a countable set of families of such solutions (see, e.g., [9, 10]). The most interesting of them is the solution in the form of a solitary wave, the so-called horseshoe soliton.

For an isothermal liquid film, the solitary solution was first found numerically in [11]. Soliton regimes for a Newtonian fluid quantitatively consistent with those calculated in [11] were experimentally obtained in [12, 13].

Conclusions. It follows from the equation derived in the present work that available results for isothermal Newtonian films can be used in modeling wave processes in down-flowing nonisothermal liquid films for a wide class of temperature dependences of viscosity and thermal diffusivity if the flow rates are sufficiently low. In particular, flow regimes in the form of horseshoe solitons are expected to exist in nonisothermal films.

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